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Noncommutativity of Quantum Observables

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Given a quantum logic (**L**, $\mathscr{S}$), a measure of noncommutativity for the elements of **L** was introduced by Román and Rumbos. For the special case when **L** is the lattice of closed subspaces of a Hilbert space, the noncommutativity between two atoms of **L** was related to the transition probability between their corresponding pure states. Here we generalize this result to the case where one of the elements of **L** is not necessarily an atom.

1. PRELIMINARIES

Most of the following definitions are well known. The reader is referred to Beltrametti and Cassinelli (1981) and Jauch (1973) for further details. A complete orthocomplemented lattice ($\mathbf{L}$, $\leqslant$, $\wedge$, $\vee$, $^{\perp}$) is said to be *orthomodular* if, given $a \leqslant b$ in **L**, then $b = a \vee (b \wedge a^{\perp})$. A map $s: \mathbf{L} \to [0, 1]$ is a *state* on **L** if $s(0) = 0$, $s(1) = 1$, and $s(\vee a_i) = \sum s(a_i)$ given $a_i \leqslant a_j^{\perp}$ for $i \neq j$. Here 1 and 0 also denote, respectively, the greatest and least elements of **L**. A set $\mathscr{S}$ of states is *full* whenever $s(a) \leqslant s(b)$ for all $s \in \mathscr{S}$ implies $a \leqslant b$. Moreover, a state s is *pure* if it cannot be expressed as a convex combination of other elements of $\mathscr{S}$. A pair (**L**, $\mathscr{S}$) where **L** is an orthomodular lattice and $\mathscr{S}$ is a full set of states is generally known in the literature as a *quantum logic.*

Let $\mathscr{B}(\mathbb{R})$ denote, as usual, the Borel sets of $\mathbb{R}$. An **L**-*observable* (or observable for short when no confusion arises) is just an **L**-valued measure, that is, a map $\mathfrak{O}: \mathscr{B}(\mathbb{R}) \to \mathbf{L}$ satisfying $\mathfrak{O}(\varnothing) = 0$, $\mathfrak{O}(\mathbb{R}) = 1$, and $\mathfrak{O}(\bigcup B_i) = \sum \mathfrak{O}(B_i)$ given $B_i \cap B_j = \varnothing$ when $i \neq j$.

Given an orthomodular lattice **L**, Román and Rumbos (1991) propose the use of a noncommutative "conjunction" in **L**, denoted by the amper-

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sand & and defined by $a \& b = (a \vee b^{\perp}) \wedge b$ for any $a, b \in \mathbf{L}$. Here one readily recognizes the Sasaki projection as the map $(\_) \& b$. It is well known that this map preserves arbitrary unions. It will always be assumed here that the lattice **L** is atomic (and hence atomistic) and has the so-called *covering property*, that is (in one of its equivalent formulations), if $a, p \in \mathbf{L}$ so that p is an atom and $p \nleq a^{\perp}$, then $p \& a$ is an atom. These two properties are usually taken for granted when speaking about quantum logics.

In Román and Rumbos (1991) the commutativity gap between any two elements $a, b \in \mathbf{L}$ is defined by $\Delta(a, b) = \sup_{s \in \mathcal{S}} |s(a \& b) - s(b \& a)|$. This definition can be extended to arbitrary **L**-observables as suggested in Maczynski (1981) and Rumbos (1993) as follows:

If $\mathfrak{O}$, $\mathfrak{P}$ are two **L**-observables, then

$$\Delta(\mathfrak{O}, \mathfrak{P}) \sup_{E, F \in B(R)} \Delta(\mathfrak{O}(E), \mathfrak{P}(F))$$

In, Rumbos (1993) it was seen that whenever there exists a bijection between the atoms of **L** and the pure states of $\mathcal{S}$, one can define the concept of transition probability in the quantum logic $(\mathbf{L}, \mathcal{S})$ in the following way:

If s_a and s_b are two pure states corresponding to the atoms a and b in **L**, the transition probability $\operatorname{trp}(s_a, s_b)$ between s_a and s_b is given by

$$\operatorname{trp}(s_a, s_b) = \begin{cases} 1 - \Delta^2(a, b) & \text{if } a \nleq b^{\perp} \\ 0 & \text{if } a \leq b^{\perp} \end{cases}$$

This definition was motivated from the case $\mathbf{L} = \mathcal{P}(\mathbf{H})$, where $\mathcal{P}(\mathbf{H})$ is the lattice of closed subspaces (or equivalently the lattice of projections) of the Hilbert space $(\mathbf{H}, \langle \cdot, \cdot \rangle)$; here $\langle \cdot, \cdot \rangle$ is, as usual, the scalar product. If $\mathcal{U}$ denotes the set of unit vectors of **H**, a full set of (pure) states is given by

$$\mathcal{S} = \{s_u : \mathcal{P}(\mathbf{H}) \to [0, 1] \mid s_u(p) = \langle p(u), u \rangle \ \forall p \in \mathbf{L}, u \in \mathcal{U}\}$$

The transition probability between s_u and s_v is given in the usual way by $|\langle u, v \rangle|^2$. If p_u denotes the one-dimensional projection onto the space generated by $u \in \mathcal{U}$, then $p_u \leftrightarrow s_u$ is a one-to-one correspondence between the atoms of $\mathcal{P}(\mathbf{H})$ and the pure states of $\mathcal{S}$. The following proposition was proved in Román and Rumbos (1991).

Proposition 1.1. Given $u, v \in \mathcal{U}$, $\langle u, v \rangle \neq 0$, then

$$\Delta(p_u, p_v) = (1 - |\langle u, v \rangle|^2)^{1/2}$$

It is clear from here how to obtain the more general definition of transition probability as given above.

It is well known that the spectral theorem yields a bijection between $\mathscr{P}(\mathbf{H})$-observables and self-adjoint operators on $\mathbf{H}$. The eigenvalues of the operator are the possible values of the observable. When $\mathbf{L}=\mathscr{P}(\mathbf{H})$ and $\mathfrak{O}$, $\mathfrak{P}$ are observables with pure point spectra and nondegenerate eigenvalues, it is straightforward from the definition of $\Delta(\mathfrak{O}, \mathfrak{P})$ and Proposition 1 that if $\{\varphi_i\}$ and $\{\psi_j\}$ are, respectively, the discrete sets of eigenstates of $\mathfrak{O}$ and $\mathfrak{P}$, then

$$\Delta(\mathfrak{O}, \mathfrak{P}) = \sup_{i,j} (1 - |\langle \varphi_i, \psi_j \rangle|^2)^{1/2}$$

Now, what if one of the eigenvalues of $\mathfrak{O}$ was degenerate and possessed an eigenspace of dimension different from 1? Or what if $\mathfrak{O}$ has a continuous spectrum? Would a similar result hold? We shall presently see that this is indeed the case.

2. THE MAIN RESULT

In this section $\mathbf{L}=\mathscr{P}(\mathbf{H})$, $\mathscr{S}$ is the usual full set of (pure) states, and $\mathscr{U}$ is the set of unit vectors of $\mathbf{H}$ as described before. The properties stated in the next lemma are well known, but for the sake of completness, proofs are included.

Lemma 2.1. Let V be any closed subspace of $\mathbf{H}$ and $u \in \mathscr{U}$, $u \notin V^{\perp}$. If p_V and p_u denote, respectively, the projections onto V and the one-dimensional subspace generated by u, the following hold:

(i) $p_V \,\&\, p_u = p_u$

(ii) $p_u \,\& p_V = p_w$, where $w = p_V(u)/\|p_v(u)\|$

Proof. Part (i) is clear, since $0 \neq p_V \,\&\, p_u \leqslant p_u$ and p_u is an atom. For part (ii), observe that $p_w \leqslant p_V$ and $p_w \leqslant p_u \vee p_V^{\perp}$; from here we have that $0 \neq p_w \leqslant (p_u \vee p_V^{\perp}) \wedge p_V = p_u \,\&\, p_V$, but from the covering property $p_u \,\&\, p_V$ is an atom, so we must have $p_u \,\&\, p_V = p_w$ as stated. ■

The next corollary is an immediate consequence of the above and the fact that $(\_)\,\&\, p$ preserves joins for any $p \in \mathscr{P}(\mathbf{H})$; it gives us an explicit description of the ampersand.

Corollary 2.2. Let V and W be closed subspaces of $\mathbf{H}$, and let $\{v_1, v_2, \ldots\}$ be an orthonormal basis (not necessarily finite) for V. The following identity then holds:

$$p_V \mathbin{\&} p_W = \bigvee p_{w_i}, \qquad \text{where} \quad w_i = \frac{p_W(v_i)}{\|p_W(v_i)\|}$$

We are now ready to state the main result.

Theorem 2.3. Let V be a closed subspace of $\mathbf{H}$ and $w \in \mathscr{U}$, $w \notin V^\perp$. Then

$$\Delta(p_w, p_V) = (1 - \|p_V(w)\|^2)^{1/2}$$

Proof. First observe that

$$\Delta(p_w, p_V) = \sup_{u \in U} |\langle (p_V \mathbin{\&} p_w - p_w \mathbin{\&} p_V)\, u, u\rangle| = \|p_V \mathbin{\&} p_w - p_w \mathbin{\&} p_V\|$$

where by abuse of notation $\|\cdot\|$ will also denote the operator norm.

If $w \in V$, then $\|p_V(w)\| = 1$ and $\Delta(p_w, p_V) = \|p_w - p_w\| = 0$, so the result clearly holds. Suppose now that $w \notin V$. From the definition of Δ and the lemma we have $\Delta(p_w, p_V) = \|p_V \mathbin{\&} p_w - p_w \mathbin{\&} p_V\| = \|p_w - p_\psi\|$, where

$$\psi = \frac{p_V(w)}{\|p_V(w)\|}$$

so that

$$\begin{aligned}
\Delta^2(p_w, p_V) &= \|p_w - p_\psi\|^2 \\
&= \|(p_w - p_\psi)(p_w - p_\psi)\| \\
&= \|p_w + p_\psi - p_w p_\psi - p_\psi p_w\| \\
&= \sup_{u \in U} |\langle (p_w + p_\psi - p_w p_\psi - p_\psi p_w)\, u, u\rangle| \qquad (1)
\end{aligned}$$

Given any $u \in \mathscr{U}$ and noting that

$$\langle w, \psi\rangle = \left\langle w, \frac{p_V(w)}{\|p_V(w)\|}\right\rangle = \|p_V(w)\|$$

one has

$$\begin{aligned}
&\langle (p_w + p_\psi - p_w p_\psi - p_\psi p_w)\, u, u\rangle \\
&\quad = \langle\langle w, u\rangle\, w + \langle \psi, u\rangle\, \psi - \langle p_V(w), u\rangle\, w - \langle w - \langle w, u\rangle\, p_V(w), u\rangle \\
&\quad = \left\langle \langle w, u\rangle [w - p_V(w)] - \langle p_V(w), u\rangle \left(w - \frac{p_V(w)}{\|p_V(w)\|^2}\right), u \right\rangle \\
&\quad = \left\langle \langle w - p_V(w), u\rangle [w - p_V(w)] \right. \\
&\qquad \left. + [1 - \|p_V(w)\|^2] \langle p_V(w), u\rangle \frac{p_V(w)}{\|p_V(w)\|^2}, u \right\rangle \\
&\quad = |\langle w - p_V(w), u\rangle|^2 + (1 - \|p_V(w)\|^2) \left| \left\langle \frac{p_V(w)}{\|p_V(w)\|}, u \right\rangle \right|^2 \qquad (2)
\end{aligned}$$

Using the fact that $[w - p_V(w)]/\|w - p_V(w)\|$ and $p_V(w)/\|p_V(w)\|$ are part of an orthonormal basis, when u is expressed in terms of this basis we obtain

$$1 = \|u\| \geqslant \left| \left\langle \frac{w - p_V(w)}{\|w - p_V(w)\|}, u \right\rangle \right|^2 + \left| \left\langle \frac{p_V(w)}{\|p_V(w)\|}, u \right\rangle \right|^2$$

Observing that $\|w - p_V(w)\|^2 = 1 - \|p_V(w)\|^2$, we combine the above with expression (2) in order to get

$$|\langle w - p_V(w), u\rangle|^2 + [1 - \|p_V(w)\|^2] \left| \left\langle \frac{P_V(w)}{\|p_V(w)\|}, u \right\rangle \right|^2 \leqslant 1 - \|p_V(w)\|^2$$

Since this holds for any $u \in \mathscr{U}$, expression (1) is also bounded above by $1 - \|p_V(w)\|^2$. Noting that for $u = w$ this upper bound is actually attained, we conclude that $\|p_V \,\&\, p_w - p_w \,\&\, p_v\|^2 = 1 - \|p_v(w)\|^2$ and hence $\Delta(p_w, p_v) = 1 - \|p_V(w)\|^2]^{1/2}$, which is the desired result. ■

Corollary 2.4. Let $\mathscr{S} = \{s_u : \mathscr{P}(\mathbf{H}) \to [0, 1] \mid u \in \mathscr{U}\}$ be the usual full set of states on $\mathscr{P}(\mathbf{H})$. For any $p \in \mathscr{P}(\mathbf{H})$ we have that

$$s_u(p) = \begin{cases} 1 - \Delta^2(p_u, p) & \text{if } \; p_u \nleqslant p^\perp \\ 0 & \text{if } \; p_u \leqslant p^\perp \end{cases}$$

Proof. Immediate, since $s_u(p) = \|p(u)\|^2$. ■

Corollary 2.5. Let $\mathfrak{D}$ and $\mathfrak{P}$ be two $\mathfrak{P}(\mathbf{H})$-observables. If $\mathfrak{D}$ has a pure point spectrum $\{\lambda_i\}$ consisting of nondegenerate eigenvalues and $\{v_i\}$

is the corresponding set of eigenvectors, the measure of noncommutativity between $\mathfrak{O}$ and $\mathfrak{P}$ is given by

$$\Delta(\mathfrak{O}, \mathfrak{P}) = \sup_{\substack{E \in B(R) \\ i \in \mathbb{N}}} [1 - s_{V_i}(\mathfrak{P}(E))]^{1/2}$$

Proof. Immediate from the definition of $\Delta(\mathfrak{O}, \mathfrak{P})$ and Corollary 2.4. ∎

To conclude, we just point out that the proof of Theorem 2.3 avoids the use of matrices, as in Maczynski (1981) and Román and Rumbos (1991); this has the advantage that the closed subspace V can be taken to be infinite dimensional.

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